SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
BHASKARA CONTEST - FINAL - JUNIOR
CLASS - IX & X

Instructions:

1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

1. (a) Find all prime numbers p such that $4p^2 + 1$ and $6p^2 + 1$ are also primes.

(b) Determine real numbers x, y, z, u such that

$xyz + xy + yz + zx + x + y + z = 7$

$yzu + yz + zu + uy + y + z + u = 9$

$zux + zu + ux + xz + z + u + x = 9$

$uxy + ux + xy + yu + u + x + y = 9$

Sol. (a) First consider primes P = 2, 3, 5

P = 2, $4p^2 + 1 = 17$ is prime but $6P^2 + 1 = 25$ is not

P = 3, $4p^2 + 1 = 37$ is prime but $6P^2 + 1 = 55$ is not

P = 5, $4p^2 + 1 = 101$ is prime & $6P^2 + 1 = 151$ is also prime

let P be prime greater than 5.

$4P^2 + 1 = 5P^2 - (P^2 - 1) \equiv -(P^2 - 1) \pmod 5$

$6P^2 + 1 = 5(P^2 - P - 1) + (P + 2)(P + 3) \equiv (P + 2)(P + 3) \pmod 5$

$\& -(P - 1)(P)(P + 1)(P + 2)(P + 3) \equiv 0 \pmod 5$

$\therefore P(4P^2 + 1)(6P^2 + 1) \equiv 0 \pmod 5$

then $(4P^2 + 1)(6P^2 + 1) \equiv 0 \pmod 5$

because P & 5 are coprime.

Hence $4P^2 + 1$ & $6P^2 + 1$ is a compost number.

Because each is greater than 5 and one of them is divisible by 5.

$\therefore$ only 1 solution, $\boxed{P = 5}$

(b) Add 1 both sides in each equations.

$(x + 1)(y + 1)(z + 1) = 8$ .....(1)

$(y + 1)(z + 1)(u + 1) = 10$ .....(2)

$(z + 1)(u + 1)(y + 1) = 10$ .....(3)

$(u + 1)(x + 1)(y + 1) = 10$ .....(4)

$[(x + 1)(y + 1)(z + 1)(u + 1)]^3 = 8000 = (20)^3$

$(x + 1)(y + 1)(z + 1)(u + 1) = 20$ .....(5)

(5) ÷ (1)

$$\frac{(x+1)(y+1)(z+1)(u+1)}{(x+1)(y+1)(z+1)} = \frac{20}{8} = \frac{5}{2}$$

$u + 1 = \frac{5}{2}$ $u = \frac{5}{2} - 1 = \frac{3}{2}$, $\therefore \boxed{u = \frac{3}{2}}$

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(5) ÷ (2)

$y + 1 = \frac{20}{10} \quad \Rightarrow x + 1 = 2 \quad \Rightarrow \quad x = 1$

(5) ÷ (3)

$y + 1 = \frac{20}{10} = 2 \quad \Rightarrow y = 1$

(5) ÷ (4)

$z + 1 = \frac{20}{10} = 2 \quad \Rightarrow z = 1$

$\therefore$ x, y, z, u is $\boxed{1,1,1,\frac{3}{2}}$

2. If x, y, z, p, q, r are distinct real numbers such that

$$\frac{1}{x+p}+\frac{1}{y+p}+\frac{1}{z+p}=\frac{1}{p}$$

$$\frac{1}{x+q}+\frac{1}{y+q}+\frac{1}{z+q}=\frac{1}{q}$$

$$\frac{1}{x+r}+\frac{1}{y+r}+\frac{1}{z+r}=\frac{1}{r}$$

find the numerical value of $\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)$

Sol. Let p, q, r are roots of cubic in t

$\therefore \quad \frac{1}{x+t}+\frac{1}{y+t}+\frac{1}{z+t}=\frac{1}{t}$

$\Rightarrow \quad \frac{1}{x+t}+\frac{1}{y+t}=\frac{1}{t}-\frac{1}{z+t}$

$\Rightarrow \quad \frac{y+t+x+t}{(x+t)(y+t)}=\frac{z+t-t}{t(z+t)}$

$\Rightarrow$ (x + y + 2t)(t)(z + t) = z(x + t)(y + t)

$\Rightarrow$ $(tx + ty + 2t^2)(z + t) = z(x + t)(y + t)$

$\Rightarrow$ $tzx + tzy + 2t^2z + t^2x + t^2y + 2t^3 = z(xy + xt + yt + t^2)$

$2t^3 + t^2(2z + x + y - z) + (zx + zy - zx - zy)t - zxy = 0$

$\Rightarrow$ $2t^3 + (x + y + z)t^2 - xyz = 0$

This equation is cubic in t whose roots are p, q, r.

$$2\left(\frac{1}{t}\right)^3 + (x + y + z)\left(\frac{1}{t}\right)^2 - xyz = 0$$

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This equation is cubic in $\frac{1}{t}$ whose roots are $\frac{1}{p},\frac{1}{q},\frac{1}{r}$.

$\Rightarrow \quad \frac{2}{t^3}+\frac{x+y+z}{t^2} - xyz = 0$

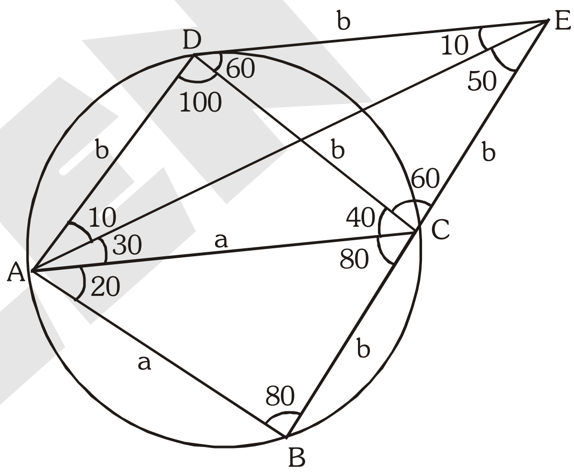
$-xyzt^3 + (x + y + z)t + 2 = 0$

$xyzt^3 - (x + y + z)t - 2 = 0$

sum of zeros, i.e. $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 0$

3. ADC and ABC are triangles such that AD = DC and CA = AB. If $\angle CAB = 20°$ and $\angle ADC = 100°$, without using trigonometry, prove that AB = BC + CD.

Sol. Since $\angle ADC = 100$



$\angle CAB = 20$

$\angle ABC = \angle ACB = 80$

$\angle ADC + \angle ABC = 180$

So ABCD is a cyclic quadrilateral.

Produce BC to E such that CE = CD

Since $\angle DCB = 40 + 80 = 120$

$\therefore \angle DCE = 60$

$\therefore \Delta CDE$ is an equilateral Δ.

In ΔADE, $\angle ADE = 100 + 60 = 160$

AD = DE = b

$\therefore \angle AED = \angle DAE = 10°$

So $\angle AEB = 50°$

In ΔABE

$\angle BAE = 50°$

$\therefore$ AB = BE

AB = BC + CE

AB = BC + CD

4. (a) a, b, c, d are positive real numbers such that abcd = 1. Prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geq 4$$

(b) In a scalene triangle ABC, $\angle BAC = 120°$. The bisectors of the angles A, B ande C meet the opposite sides in P, Q and R respectively. Prove that the circle on QR as diameter passes through the point P.

Sol. **(a)** abcd = 1

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+\frac{1}{ab}}{1+c}+\frac{1+\frac{1}{bc}}{1+d}$$

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$\Rightarrow \frac{1+ab}{1+a}+\frac{1+ab}{ab+abc}+\frac{1+bc}{1+b}-\frac{1+bc}{bc+bcd}$

$\Rightarrow (1+ab)\left[\frac{1}{1+a}+\frac{1}{ab+abc}\right]+(1+bc)\left[\frac{1}{1+b}+\frac{1}{bc+bcd}\right]$

$\Rightarrow \geq (1+ab)\left[\frac{4}{1+a+ab+abc}\right]+(1+bc)\left[\frac{4}{1+b+bc+bcd}\right]$

($\because$ by Titu's formula $\frac{1^2}{1+a}+\frac{1^2}{ab+abc}\geq\frac{(1+1)^2}{1+a+ab+abc}\geq\frac{4}{1+a+ab+abc}$)

$\Rightarrow \geq 4\left[\frac{1+ab}{1+a+ab+abc}+\frac{1+bc}{1+b+bc+bcd}\right]$

$\Rightarrow \geq 4\left[\frac{1+ab}{1+a+ab+abc}+\frac{a+abc}{a+ab+abc+abcd}\right]$

$\Rightarrow \geq 4\left[\frac{1+ab}{1+a+ab+abc}+\frac{a+abc}{1+a+ab+abc}\right]$

$\Rightarrow \geq 4\left[\frac{1+ab+a+abc}{1+a+ab+abc}\right]$

$\Rightarrow \geq 4$

(b) Extend BA to X and CA to Y

So $\angle CAX = 60$ and $\angle BAY = 60$

In ΔABP, AC is the exterior angle bisector and BQ is the interior angle bisector so Q is the excenter of ΔABP

So QP bisects $\angle APC$.

(In a triangle two external angle bisectors and one internal bisector are always concurrent)

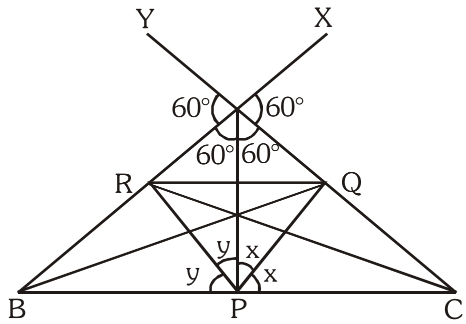
Similarly PR bisects $\angle BPA$

$\therefore 2x + 2y = 180°$

$x + y = 90$

$\angle QPR = 90°$

Circle on QR as diameter passes through the point P.



5. (a) Prove that $x^4 + 3x^3 + 6x^2 + 9x + 12$ can not be expressed as a product of two polynomials of degree 2 with integer coefficients.

(b) 2n + 1 segments are marked on a line. Each of these segments intersects at least n other segments. Prove that one of these segments intersects all other segments.

Sol. **(a)** Let $x^4 + 3x^3 + 6x^2 + 9x + 12 = (x^2 + ax + b)(x^2 + cx + d)$

where a, b, c, d are integers.

$= x^4 + cx^3 + dx^2 + ax^3 + acx^2 + adx + bx^2 + bcx + bd$

$= x^4 + (c + a)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd$

on comparing coefficients of both equations, we get

$a + c = 3$ ......(1)

$b + d + ac = 6$ ......(2)

$ad + bc = 9$ ......(3)

$bd = 12$ ......(4)

from equation (4), since b & d both are integers, we can have following possibilities.

$bd = 12 \times 1$	1×12	-12×-1	-1×-12
$= 6 \times 2$	2×6	-6×-2	-2×-6
$= 4 \times 3$	3×4	-4×-3	-3×-4

Case 1

$b = 12, d = 1$

$b + d + ac = 6$

$12 + 1 + ac = 6$

$ac = -7$

also $a + c = 3$

So equation whose roots a,c is

$x^2 - 3x - 7 = 0$

$D = 9 - 4(1) - (-7) = 37$

$D \neq$ perfect square, so no integers a and c.

Similarly b = 1 & d = 12 will also not satisfy.

Case 2

$b = 6, d = 2$

$6 + 2 + ac = 6$

$ac = -2$

& $a + c = 3$

So equation whose roots a,c is

$x^2 - 3x - 2 = 0$

$D = (-3)^2 - 4(1)(-2) = 9 + 8 = 17$

$D \neq$ perfect square, So no integer a & c.

Similarly b = 2 & d = 6 will also not satisfy.

Case 3

$b = 4, d = 3$

$4 + 3 + ac = 6$

$ac = -1$

& $a + c = 3$

So equation whose roots a,c is

$x^2 - 3x - 1 = 0$

$D = (-3)^2 - 4(1)(-1) = 9 + 4 = 13$

$D \neq$ perfect square, so no integer a & c.

Similarly b = 3, d = 4 will also not satisfy.

Case 4

$b = -12, d = -1$

$-12 - 1 + ac = 6$

$ac = 19$ & $a + c = 3$

So equation whose roots a,c is

$x^2 - 3x + 19 = 0$

$D = (-3)^2 - 4(1)(19) = 9 - 76 = -67 < 0$

No real solution & hence no integer a & c.

Similarly b = –1 & d = –12 will also not satisfy.

Case 5

$b = -6, d = -2$

$-6 - 2 + ac = 6$

$ac = 14$ & $a + c = 3$

So equation whose roots a,c is

$x^2 - 3x + 14 = 0$

$D = (-3)^2 - 4(1)(14) = 9 - 56 = -47 < 0$

No real solution & hence no integer a & c.

Similarly b = –2 & d = –6 will also not satisfy.

Case 6

$b = -4, d = -3$

$-4 - 3 + ac = 6$

$ac = 13$ & $a + c = 3$

So equation whose roots a,c is

$x^2 - 3x + 13 = 0$

$D = (-3)^2 - 4(1)(13) = 9 - 52 = -43 < 0$

No real solution & hence no integer a & c.

Similarly b = –3 & d = –4 will also not satisfy.

So the given equation cannot be expressed as a product of two polynomials of degree 2 with integer coefficients.

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(b) Let us define each line segment by interval of type [a, b]. We will be referring a as left end and b as right end.

Let us consider a set X = union of n left most (by picking left ends). The line segments and Y = union of n right most (by picking right ends) line segments.

Now in X, right most left end will be left of the left most right end otherwise there will be a line in X which will have less than n intersections. This implies each segment in X pairwise intersecting with each other which corresponds to $n - 1$ intersections of each line segments with in these n line segments in X.

Similarly in Y left most right end will be right of the right most left end otherwise there will be a line in Y which will have less than $n - 1$ intersections. This implies each segment in Y is also pairwise intersecting with each other corresponds to $n - 1$ intersections of each line segments with in these n line segments in Y.

Now if there is a any common line segment between X and Y then this line segment will intersect with all line segments of X as well as all line segments of Y. So this line segment intersect with all other line segments and we are done.

If there is no common line segment between X and Y, then X and Y will be disjoint sets having n line segments. There will be one more line segment (as there are total $2n + 1$ line segments) this line segment must intersect with each line segment contained in X as we need n intersection on each line segment similarly this line segment must intersect with each line segment in Y. So this is the line segment which intersects with other line segments and we are done.

6. If a, b, c, d are positive real numbers such that $a^2 + b^2 = c^2 + d^2$ and $a^2 + d^2 - ad = b^2 + c^2 + bc$, find the value of $\frac{ab+cd}{ad+bc}$.

Sol. Let $a^2 + b^2 = c^2 + d^2 = k^2$, $k > 0$

Let $a = a'k$; $a' > 0$

$b = b'k$; $b' > 0$

$c = c'k$; $c' > 0$

$d = d'k$; $d' > 0$

$a^2 + b^2 = c^2 + d^2 = k^2$

$\therefore a'^2 + b'^2 = 1$

and $c'^2 + d'^2 = 1$

Let $a' = \sin\alpha$, $b' = \cos\alpha$

$c' = \sin\beta$, $d' = \cos\beta$

$a^2 + d^2 - 4d = b^2 + c^2 = bc$

$k^2(\sin^2\alpha + \cos^2\beta - \sin\alpha \sin\beta) = k^2(\cos^2\alpha + \sin^2\beta + \cos\alpha\sin\beta)$

$\sin^2\alpha - \cos^2\alpha + \cos^2\beta - \sin^2\beta = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

$\sin(\alpha + \beta) \sin(\alpha - \beta) + \sin(\alpha + \beta) \sin(\alpha - \beta) = \sin(\alpha + \beta)$

$2\sin(\alpha - \beta) = 1$

$\sin(\alpha - \beta) = \frac{1}{2}$

Now, we have to find, $\frac{ab+cd}{ad+bc}$

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$$= \frac{\sin\alpha\cos\alpha + \sin\beta\cos\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$= \frac{2\sin\alpha\cos\alpha + 2\sin\beta\cos\beta}{2(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}$$

$$= \frac{\sin 2\alpha + \sin 2\beta}{2\sin(\alpha+\beta)}$$

$$= \frac{2\sin(\alpha+\beta).\cos(\alpha-\beta)}{2\sin(\alpha+\beta)}$$

$$= \cos(\alpha-\beta) = \sqrt{1-\sin^2(\alpha-\beta)}$$

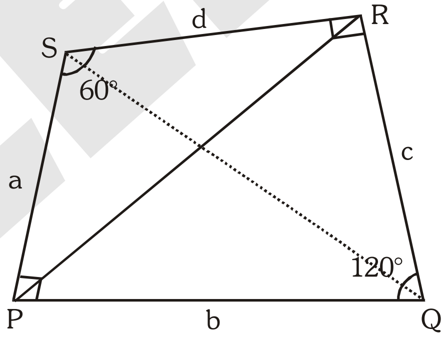
$$= \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Alternate solution

Let us consider a quadrilateral of side length a, b, c, d

such that $a^2 + b^2 = c^2 + d^2$ & $a^2 + d^2 - ad = b^2 + c^2 + bc$ .....(i)

Hence in ΔPQR.



$\Rightarrow \cos 20° = \frac{b^2 + c^2 - PR^2}{2bc}$

$\Rightarrow -\frac{1}{2} = \frac{b^2 + c^2 - PR^2}{2bc}$

$\Rightarrow PR^2 = b^2 + c^2 + bc$ ....(ii)

& in ΔPRS,

$\cos 60° = \frac{a^2 + d^2 - PR^2}{2.ad}$

$PR^2 = a^2 + d^2 - ad$ ......(iii)

By (i), (ii) & (iii), we can say that our construction of quadrilateral is correct.

Also, when $\angle P = 90° \Rightarrow SQ^2 = a^2 + b^2$
& $\angle R = 90° \Rightarrow SQ^2 = c^2 + d^2$ {which are equal as given in questions}

Now [PQRS] = [PQR] + [PRS] = $\frac{1}{2}$. sin 120°.bc + $\frac{1}{2}$ .sin 60° . ad

[PQRS] = $\frac{\sqrt{3}}{4}$(ad + bc) ......(iv)

Also [PQRS] = [PQS] + [SRQ]

[PQRS] = $\frac{1}{2}$(ab + cd) .....(v)

So, by (iv) & (v) $\frac{ab+cd}{ad+bc} = \frac{\sqrt{3}}{2}$

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