SOLUTION
THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA
KAPREKAR CONTEST - FINAL - SUB JUNIOR
CLASS - VII & VIII

Instructions:
1. Answer as many questions as possible.
2. Elegant and novel solutions will get extra credits.
3. Diagrams and explanations should be given wherever necessary.
4. Fill in FACE SLIP and your rough working should be in the answer book.
5. Maximum time allowed is THREE hours.
6. All questions carry equal marks.

1. (a) Find all three digit numbers in which any two adjacent digits differ by 3.

(b) There are 5 cards. Five positive integers (may be different or equal) are written on these cards, one on each card. Abhiram finds the sum of the numbers on every pair of cards. He obtains only three different totals 57, 70, 83. Find the largest integer written on a card.

Ans. (a) 236 (b) 48

Sol. (a) Let the three digit number be abc

Case I

a	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*
b	3	4	5	6	7	8	9
c	0	1	2	3	4	5	6

Total numbers possible = 9 ×7 = 63

Case II

a	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*	*1 to 9*
b	0	1	2	3	4	5	6
c	3	4	5	6	7	8	9

Total numbers possible = 9 ×7 = 63

Case III

	a	1	2	3	4	5	6
	b	4	5	6	7	8	9
Not Possible	***c***	(1, 7)	(2, 8)	(3,9)	4	5	6
Possible value		8 Values	8 Values	8 Values	9 Values	9 Values	9 Values

Total possibilities = 8 × 3 + 9 ×3 = 24 + 27 ⇒ 51

Case IV

	a	9	8	7	6	5	4	3
	b	6	5	4	3	2	1	0
Not Possible	***c***	9,6	8,2	7,1	6,0	5	3	3
Possible value		8 Values	8 Values	8 Values	8 Values	9 Values	9 Values	9 Values

Total possibilities = 8 × 4 + 9 ×3 = 59

Total numbers possible = 63 + 63 + 51 + 59 = 236

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(b) There are five numbers (same or different) one on each card.

Three different sums are 57,70,83

So possibility of numbers are a, a, a, b, c

$a + b = 57$, $b + c = 70$, $c + a = 83$

$\Rightarrow 2a = 70$, $a = 35$

$b = 57 - 35 = 22$

$c = 70 - 22 = 48$

Largest integer is 48.

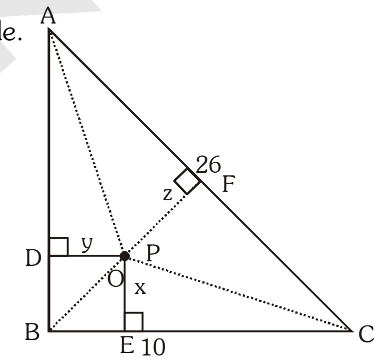
2. (a) ABC is a triangle in which AB = 24, BC = 10 and CA = 26. P is a point inside the triangle. Perpendiculars are drawn to BC, AB and AC. Length of these perpendiculars respectively are x, y and z. Find the numerical value of $5x + 12y + 13z$.

(b) If $x^2(y + z) = a^2$, $y^2(z + x) = b^2$, $z^2(x + y) = c^2$, $xyz = abc$

prove that $a^2 + b^2 + c^2 + 2abc = 1$

Ans. (a) 20

Sol. (a) (24,10,26) is a phythagorean triplet. So ABC is a right angled triangle.



ar. (ΔABC) = ar. (ΔAOB) + ar. (ΔBOC) + ar. (ΔAOC)

$\Rightarrow \frac{1}{2} \times 24 \times 10$

$= \frac{1}{2} \times 10 \times x + \frac{1}{2} \times 24 \times y + \frac{1}{2} \times 26 \times z$

$\Rightarrow 120 = 5x + 12y + 13z$

(b) $x^2 (y + z) = a^2$,

$y^2 (x + z) = b^2$, $z^2 (x + y) = c^2$

$xyz = abc$

Multiply the given three equations.

$x^2y^2z^2 (x + y) (y + z) (z + x) = a^2b^2c^2$

$\Rightarrow$ $(x + y) (y + z) (z + x) = 1$ $(xyz = abc)$

$\Rightarrow$ $(xy + xz + y^2 + yz) (z + x) = 1$

$\Rightarrow$ $(xyz + x^2y + xz^2 + x^2z + y^2z + y^2x + yz^2 + xyz = 1$

$\Rightarrow$ $x^2y + xy^2 + xz^2 + x^2z + y^2z + yz^2 + 2xyz = 1$ ...(1)

$\Rightarrow$ To Prove : $a^2 + b^2 + c^2 + 2abc = 1$

$\Rightarrow$ $x^2(y + z) + y^2 (x + z) + z^2 (x + y) + 2xyz$

$\Rightarrow$ $x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + 2xyz$ ...(2)

Equation (1) gives value of (2) as 1

Hence proved.

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3. If $X = \frac{a^2-(2b-3c)^2}{(3c+a)^2-4b^2}+\frac{4b^2-(3c-a)^2}{(a+2b)^2-9c^2}+\frac{9c^2-(a-2b)^2}{(2b+3c)^2-a^2}$

$Y = \frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4x)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}$

find 2017 (X + Y)

Ans. 4034

Sol. $X = \frac{a^2-(2b-3c)^2}{(3c+a)^2-4b^2}+\frac{4b^2-(3c-a)^2}{(a+2b)^2-9c^2}+\frac{9c^2-(a-2b)^2}{(2b+3c)^2-a^2}$

$X = \frac{(a-2b+3c)(a+2b-3c)}{(a-2b+3c)(a+2b+3c)}+\frac{(2b-3c+a)(2b+3c-a)}{(a+2b-3c)(a+2b+3c)}+\frac{(3c-a+2b)(3c+a-2b)}{(2b+3c-a)(2b+3c+a)}$

$X = \frac{(a+2b-3c)}{a+2b+3c}+\frac{(-a+2b+3c)}{a+2b+3c}+\frac{(a-2b+3c)}{a+2b+3c}$

$X = \frac{a+2b-3c-a+2b+3c+a-2b+3c}{a+2b+3c} = \frac{a+2b+3c}{a+2b+3c} = 1$

$X = 1$

$Y = \frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9x^2}$

$Y = \frac{(3y-4z+2x)(3y+4z-2x)}{(2x+3y-4z)(2x+3y+4z)}+\frac{(4z-2x+3y)(4z+2x-3y)}{(3y+4z-2x)(3y+4z+2x)}+\frac{(2x-3y+4z)(2x+3y-4z)}{(4z+2x-3y)(4z+2x+3y)}$

$Y = \frac{(-2x+3y+4z)}{2x+3y+4z}+\frac{2x-3y+4z}{2x+3y+4z}+\frac{2x+3y-4z}{2x+3y+4z}$

$Y = \frac{-2x+3y+4z+2x-3y+4z+2x+3y-4z}{2x+3y+4z} = \frac{2x+3y+4z}{2x+3y+4z} = 1$

$Y = 1$

$\Rightarrow$ 2017 (X + Y) = 2017 (1 + 1) = 2017 × 2 = 4034

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4. The sum of the ages of a man and his wife is six times the sum of the ages of their children. Two years ago the sum of their ages was ten times the sum of the ages of their children. Six years hence the sum of their ages will be three times the sum of the ages of their children. How many children do they have?

Ans.

Sol. Let sum of age of man and wife be x years

Sum of ages of their chidredn b y years

Number of children be n

$\Rightarrow$ x = 6y ...(1)

Two years ago

(x – 4) = 10 (y – 2n)

x – 10 y = 4 – 20n ...(2)

Six years hence,

x + 12 = 3(y + 6n)

x + 12 = 3y + 18 n ...(3)

Solving equations (1), (2) and (3)

6y – 10y = 4 – 20 n

$\Rightarrow$ 20 n = 4 + 4 y

5n = 1 + y ...(4)

$\Rightarrow$ 6y + 12 = 3y + 18 n

$\Rightarrow$ 3y + 12 = 18 n

$\Rightarrow$ y + 4 = 6 n ...(5)

By (5) and (4)

6n – 5n = y + 4 – y – 1

n = 3

Number of children is 3.

5. (a) a, b, c are three natural numbers such that $a \times b \times c = 27846$. If $\frac{a}{6} = b + 4 = c - 4$, find a + b + c.

(b) ABCDEFGH is a regular octagon with side length equal to a. Find the area of the trapezium ABDG.

Ans.

Sol. (a) $a \times b \times c = 27846$

$\frac{a}{6} = b + 4 = c - 4 = k$

a must be a multiple of 6

$\Rightarrow$ let a = 6k

$\Rightarrow$ b = k – 4 and c = k + 4

6k (k + 4) (k – 4) = 27846

$(k - 4) \times k \times (k + 4) = 4641$

$(k - 4) \times k \times (k + 4) = 13 \times 17 \times 21$

$\Rightarrow$ k = 17

a + b + c = 6k + k – 4 + k + 4

= 8k = 136

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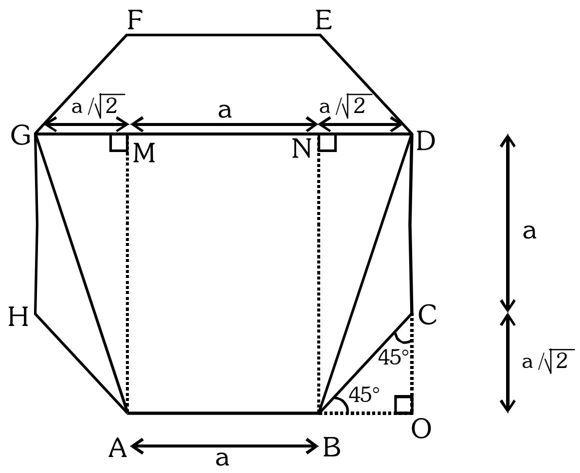
(b)



Each interior angle = $\frac{(8-2)180}{8}$ = 135°

AB = a, DG = a + $\frac{2a}{\sqrt{2}}$

Height, AM = a + $\frac{a}{\sqrt{2}}$

Area of trapezium = $\frac{1}{2}$ (Sum of parallel sides) (Distance between them)

Area = $\frac{1}{2}\left(a+\frac{a}{\sqrt{2}}\right)\left(a+a+\frac{2a}{\sqrt{2}}\right)$

= $\left(a+\frac{a}{\sqrt{2}}\right)^2$ = $a^2+\frac{a^2}{2}$ + 2.a . $\frac{a}{\sqrt{2}}$

⇒ $\frac{2\sqrt{2}a^2+\sqrt{2}a^2+4a^2}{2\sqrt{2}}$

⇒ $\frac{4+3\sqrt{2}}{2\sqrt{2}}a^2$

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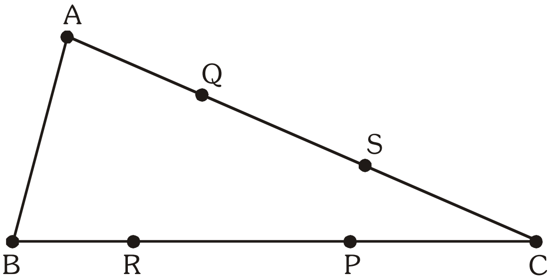
6. (a) If a, b, c are positive real numbers such that no two of them are equal, show that

$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)$

is always positive.

(b) In the figure below, P, Q, R, S are points on the sides of the triangle ABC such that

$CP = PQ = QB = BA = AR = RS = SC$



Find the $\angle C$.

Ans.

Sol. (a) Let $a > b > c$

$\Rightarrow$ $a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)$

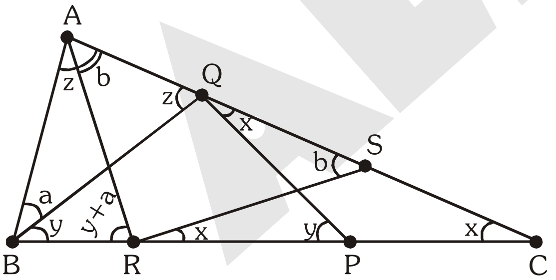
$\Rightarrow$ $(a-b)[a^2 - ac - b^2 + bc] + c(c-a)(c-b)$

$\Rightarrow$ $(a-b)[(a-b)(a+b) - c(a-b)] + c(c-a)(c-b)$

$\Rightarrow$ $\underbrace{(a-b)^2[a+b-c]}_{\text{Always Positive}} + \underbrace{c(c-a)(c-b)}_{\substack{c-a<0 \\ c-b<0 \\ \text{and given } c>0 \\ \text{It will become positive}}}$

$\therefore$ So given expression is always positive.

(b)



$QP = PC$, So $\angle PCQ = \angle PQC$ let $\angle PCQ = \angle PQC = x$

given $PQ = QB$, so $\angle PBQ = \angle BPQ = y$

$QB = AB$, So $\angle BAQ = \angle BQA = z$

$AB = AR$, So $\angle ABR = \angle ARB$

$\angle ABQ + \angle ABR = \angle ARB$

Let $\angle ABQ = a \Rightarrow a + y = \angle ARB$

$AR = RS$, So $\angle RAS = \angle RSA = b$

$RS = SC$ so $\angle SRC = \angle SCR = x$

Now by exterior angle property In ΔQPC

$y = 2x$ … (1)

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Exterior angle property In ΔARS,

$y + a = b + x$

from Equation (1) $2x + a = b + x$

$x + a = b$ ... (2)

exterior in ΔBQC, $z = y + x$ ... (3)

from (1) and (3) $z = 3x$ ... (4)

$\angle ARS = 180 - 2b$

$\angle ARB + \angle ARS + \angle SRC = 180$

$y + a + 180 - 2b + x = 180$

$2b = y + a + x$ ...(5)

from (5) and (1)

$2b = 3x + a$ ...(6)

equation (2) ×2 - equation (6)

$2x + 2a = 2b$

$3x + a = 2b$

$x = a$ ... (7)

Now in ΔABC, $\angle C + \angle A + \angle B = 180°$

$z + x + y + a = 180°$

from eq (1), eq. (4) , eq (7)

$3x + x + 2x + x = 180°$

$x = \frac{180°}{7}$

$\angle C = \frac{180°}{7}$